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Solution of the dispersionless Hirota equations

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Abstract. The dispersionless differential Fay identity is shown to be equivalent to a kernel expansion providing a universal algebraic characterization and solution of the dispersionless Hirota equations. Some calculations based on D-bar data of the action are also indicated.

1. Introduction

The KP hierarchy and its reductions (such as the NKdV equation) have been recognized as among the most important integrable systems in several fields of physics (cf [2–4, 6, 8, 13, 14, 16, 20, 22, 25, 28, 29, 30]). The solution of the hierarchy can be described by the so-called τ-function, and the Hirota bilinear formulae provide the governing equations for the τ function (cf [2, 3, 6, 8, 9, 12, 28, 29]). In particular the τ functions are identified as partition functions of matrix models for describing 2D gravity. The τ function structure of the hierarchy was found by the Kyoto group [12], and the essence of it may be summarized as the bilinear identity between the wavefunctions (Baker–Akhiezer functions) of the Lax operator. The bilinear identity also leads to the Fay identity, which corresponds to the Fay trisecant identity describing quasi-periodic solutions of the KP hierarchy (cf [2, 8, 28, 29]). As a quasi-classical (dispersionless) limit of the KP hierarchy, the dispersionless KP (DKP) hierarchy and various reductions thereof also play an increasingly important role in topological field theory and its connections to string theory and 2D gravity (cf [3, 4, 6, 7, 10, 13, 14, 16, 22, 25, 28, 29, 30]). There are also connections to twistor theory in the spirit of [28, 29] but we do not pursue this direction here.

In this paper, we discuss the dispersionless limit of the Hirota bilinear equations, which are obtained from the dispersionless (differential) Fay identity [29], for a characterization of the DKP hierarchy. We start in section 2 to provide background information on the KP and DKP hierarchies necessary for the material which follows. In particular, the dispersionless Hirota equations are formulated as the polynomial identities among the symbols with two indices F_{mn} (theorem 1). The solutions of the dispersionless Hirota equations are then given in terms of a residue formula $F_{mn} = Res_P[\lambda^m d\lambda^n_+]$ where $\lambda = P + \sum_1^\infty U_{n+1} P^{-n}$ and λ^n_+ is the polynomial part of λ^n in P. The solution F_{mn} is also identified as the second derivative of the free energy $F = \log \tau_{\mathrm{DKP}}$ with respect to the coupling constants T_m and T_n, i.e. $F_{mn} = \partial^2 F/\partial T_m \partial T_n$, which gives the two-point functions in a topological field theory (cf [3, 4, 10, 13, 14, 25]). In section 3, we present our main result (theorem 2), namely that the dispersionless (differential) Fay identity is equivalent to a kernel expansion which generates

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the dispersionless Hirota equations. Here the kernel is given by $1/\{P(\lambda)-P(\mu)\}$ where $P(\lambda)$ (or $P(\mu)$) is the inversion of λ (or μ) $= P+\sum_1^\infty U_{n+1}P^{-n}$. This kernel plays an important rule for the integrability of the DKP hierarchy (cf [22]). In section 4, we show how the Hamilton–Jacobi and hodograph analysis in [17–19, 21, 22] for the DNKdV equation yields the D-bar data in the scattering problem, and this gives a moment-type formula for the F_{mn} in terms of the D-bar data. In the DKdV situation this actually yields a direct formula for the F_{mn} providing an immediate alternative method of calculation (theorem 3).

2. Background on KP and DKP

One can begin with two pseudodifferential operators ($\partial=\partial/\partial x$)

$$L=\partial+\sum_1^\infty u_{n+1}\partial^{-n} \qquad W=1+\sum_1^\infty w_n\partial^{-n} \tag{2.1}$$

called the Lax operator and gauge operator, respectively, where the generalized Leibnitz rule with $\partial^{-1}\partial=\partial\partial^{-1}=1$ applies:

$$\partial^i f\cdot=\sum_{j=0}^\infty \binom{i}{j}(\partial^j f)\partial^{i-j}\,. \tag{2.2}$$

for any $i\in\mathbb{Z}$, and $L=W\partial W^{-1}$. The KP hierarchy then is determined by the Lax equations ($\partial_n=\partial/\partial t_n$)

$$\partial_n L=[B_n,L]:=B_nL-LB_n \tag{2.3}$$

where $B_n=L_+^n$ is the differential part of $L^n=L_+^n+L_-^n=\sum_0^\infty \ell_i^n\partial^i+\sum_{-\infty}^{-1}\ell_i^n\partial^i$. One can also express this via the Sato equation

$$\partial_n W\,W^{-1}=-L_-^n \tag{2.4}$$

which is particularly well adapted to the DKP theory. Now define the wavefunction via

$$\psi=W\,\mathrm{e}^\xi=w(t,\lambda)\mathrm{e}^\xi \qquad \xi:=\sum_1^\infty t_n\lambda^n \qquad w(t,\lambda)=1+\sum_1^\infty w_n(t)\lambda^{-n} \tag{2.5}$$

where $t_1=x$. There is also an adjoint wavefunction $\psi^*=W^{*-1}\exp(-\xi)=w^*(t,\lambda)\exp(-\xi)$, $w^*(t,\lambda)=1+\sum_1^\infty w_i^*(t)\lambda^{-i}$, and one has the equations

$$L\psi=\lambda\psi \qquad \partial_n\psi=B_n\psi \qquad L^*\psi^*=\lambda\psi^* \qquad \partial_n\psi^*=-B_n^*\psi^*\,. \tag{2.6}$$

(cf [8] for L^*, w^*, etc). Note that the KP hierarchy (2.3) is then given by the compatibility conditions among these equations with an isospectral property. Next one has the fundamental tau function $\tau(t)$ and vertex operators X, X^* satisfying

$$\begin{aligned}\psi(t,\lambda)&=\frac{X(\lambda)\tau(t)}{\tau(t)}=\frac{\mathrm{e}^\xi G_-(\lambda)\tau(t)}{\tau(t)}=\frac{\mathrm{e}^\xi\tau(t-[\lambda^{-1}])}{\tau(t)}\\ \psi^*(t,\lambda)&=\frac{X^*(\lambda)\tau(t)}{\tau(t)}=\frac{\mathrm{e}^{-\xi}G_+(\lambda)\tau(t)}{\tau(t)}=\frac{\mathrm{e}^{-\xi}\tau(t+[\lambda^{-1}])}{\tau(t)}\end{aligned} \tag{2.7}$$

where $G_\pm(\lambda)=\exp(\pm\xi(\tilde\partial,\lambda^{-1}))$ with $\tilde\partial=(\partial_1,\frac12\partial_2,\frac13\partial_3,\ldots)$ and $t\pm[\lambda^{-1}]=(t_1\pm\lambda^{-1},t_2\pm\frac12\lambda^{-2},\ldots)$. One also writes

$$\mathrm{e}^\xi:=\exp\left(\sum_1^\infty t_n\lambda^n\right)=\sum_0^\infty\chi_j(t_1,t_2,\ldots,t_j)\lambda^j \tag{2.8}$$

where the χ_j are the elementary Schur polynomials, which arise in many important formulae (cf below).

We now mention the famous bilinear identity which generates the entire KP hierarchy. This has the form

$$\oint_{\infty} \psi(t,\lambda)\psi^*(t',\lambda)\,d\lambda = 0 \tag{2.9}$$

where $\oint_{\infty}(\cdot)\,d\lambda$ is the residue integral about ∞, which we also denote $Res_{\lambda}[(\cdot)d\lambda]$. Using (2.7) this can also be written in terms of tau functions as

$$\oint_{\infty} \tau(t-[\lambda^{-1}])\tau(t'+[\lambda^{-1}])\exp\left(\xi(t,\lambda)-\xi(t',\lambda)\right)\,d\lambda = 0\,. \tag{2.10}$$

This leads to the characterization of the tau function in bilinear form expressed via $(t \to t-y,\ t' \to t+y)$

$$\left(\sum_0^{\infty}\chi_n(-2y)\chi_{n+1}(\tilde{D})\exp\left(\sum_1^{\infty}y_iD_i\right)\right)\tau\cdot\tau = 0 \tag{2.11}$$

where D_i is the Hirota derivative defined as $D_j^m a\cdot b = (\partial^m/\partial s_j^m)a(t_j+s_j)b(t_j-s_j)|_{s=0}$ and $\tilde{D} = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \ldots)$. In particular, we have from the coefficients of y_n in (2.11),

$$D_1D_n\tau\cdot\tau = 2\chi_{n+1}(\tilde{D})\tau\cdot\tau \tag{2.12}$$

which are called the Hirota bilinear equations. Such calculations with vertex operator equations and residues, in the context of finite-zone situations where the tau function is intimately related to theta functions, also led historically to the Fay trisecant identity, which can be expressed generally as the Fay identity via (cf [2, 8], here 'c.p.' means cyclic permutations)

$$\sum_{\text{c.p.}}(s_0-s_1)(s_2-s_3)\tau(t+[s_0]+[s_1])\tau(t+[s_2]+[s_3]) = 0\,. \tag{2.13}$$

This can also be derived from the bilinear identity (2.10). Differentiating this in s_0, then setting $s_0 = s_3 = 0$, then dividing by s_1s_2, and finally shifting $t \to t-[s_2]$, leads to the differential Fay identity ($\partial = \partial/\partial x$),

$$\begin{aligned}&\tau(t)\partial\tau(t+[s_1]-[s_2]) - \tau(t+[s_1]-[s_2])\partial\tau(t)\\ &\quad = (s_1^{-1}-s_2^{-1})\left[\tau(t+[s_1]-[s_2])\tau(t) - \tau(t+[s_1])\tau(t-[s_2])\right]\,.\end{aligned} \tag{2.14}$$

The Hirota equations (2.12) can also be derived from (2.14) by taking the limit $s_1 \to s_2$. The identity (2.14) will play an important role later.

Now for the dispersionless theory (DKP) one can think of fast and slow variables, etc, or averaging procedures, but simply one takes $t_n \to \epsilon t_n = T_n$ ($t_1 = x \to \epsilon x = X$) in the KP equation $u_t = \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}\partial^{-1}u_{yy}$, ($y = t_2$, $t = t_3$), with $\partial_n \to \epsilon\partial/\partial T_n$ and $u(t_n) \to U(T_n)$ to obtain $\partial_T U = 3UU_X + \frac{3}{4}\partial^{-1}U_{YY}$ when $\epsilon \to 0$ ($\partial = \partial/\partial X$ now). Thus the dispersion term u_{xxx} is removed. In terms of hierarchies we write

$$L_{\epsilon} = \epsilon\partial + \sum_1^{\infty} u_{n+1}(T/\epsilon)(\epsilon\partial)^{-n} \tag{2.15}$$

and think of $u_n(T/\epsilon) = U_n(T) + O(\epsilon)$, etc. One then takes a WKB form for the wavefunction with the action S [21]:

$$\psi = \exp\left[\frac{1}{\epsilon}S(T,\lambda)\right]. \tag{2.16}$$

Now replacing ∂_n by $\epsilon\partial_n$, where $\partial_n = \partial/\partial T_n$ now, we define $P := \partial S = S_X$. Then $\epsilon^i \partial^i \psi \to P^i \psi$ as $\epsilon \to 0$ and the equation $L\psi = \lambda\psi$ becomes

$$\lambda = P + \sum_1^\infty U_{n+1} P^{-n} \qquad P = \lambda - \sum_1^\infty P_{i+1}\lambda^{-i} \tag{2.17}$$

where the second equation is simply the inversion of the first. We also note from $\partial_n\psi = B_n\psi = \sum_0^n b_{nm}(\epsilon\partial)^m\psi$ that one obtains $\partial_n S = \mathcal{B}_n(P) = \lambda_+^n$ where the subscript $(+)$ now refers to powers of P (note $\epsilon\partial_n\psi/\psi \to \partial_n S$). Thus $B_n = L_+^n \to \mathcal{B}_n(P) = \lambda_+^n = \sum_0^n b_{nm} P^m$ and the KP hierarchy goes to

$$\partial_n P = \partial \mathcal{B}_n \tag{2.18}$$

which is the DKP hierarchy (note $\partial_n S = \mathcal{B}_n \Rightarrow \partial_n P = \partial\mathcal{B}_n$). The action S in (2.16) can be computed from (2.7) in the limit $\epsilon \to 0$ as

$$S = \sum_1^\infty T_n \lambda^n - \sum_1^\infty \frac{\partial_m F}{m}\lambda^{-m} \tag{2.19}$$

where the function $F = F(T)$ (free energy) is defined by [28]

$$\tau = \exp\left[\frac{1}{\epsilon^2}F(T)\right]. \tag{2.20}$$

The formula (2.19) then solves the DKP hierarchy (2.18), i.e. $P = \mathcal{B}_1 = \partial S$ and

$$\mathcal{B}_n = \partial_n S = \lambda^n - \sum_1^\infty \frac{F_{nm}}{m}\lambda^{-m} \tag{2.21}$$

where $F_{nm} = \partial_n\partial_m F$ which play an important role in the theory of DKP.

Now following [29] we write the differential Fay identity (2.14) with $\epsilon\partial_n$ replacing ∂_n etc in the form

$$\frac{\tau(T - \epsilon[\mu^{-1}] - \epsilon[\lambda^{-1}])\tau(T)}{\tau(T - \epsilon[\mu^{-1}])\tau(T - \epsilon[\lambda^{-1}])} = 1 + \frac{\epsilon\partial}{\mu - \lambda}\left[\log\left(\tau(T - \epsilon[\mu^{-1}])\right) - \log\left(\tau(T - \epsilon[\lambda^{-1}])\right)\right] \tag{2.22}$$

(in (2.14) take $t \to t - [s_1]$, $s_1 = \mu^{-1}$, $s_2 = \lambda^{-1}$ and insert ϵ at the appropriate places; note T is used in (2.22)). One notes from (2.8) that $\exp(\xi(\tilde{\partial}, \lambda^{-1})) = \sum_0^\infty \chi_j(\tilde{\partial})\lambda^{-j}$, so taking logarithms in (2.22) and using (2.20) yields

$$\frac{1}{\epsilon^2}\sum_{m,n=1}^\infty \mu^{-m}\lambda^{-n}\chi_n(-\epsilon\tilde{\partial})\chi_m(-\epsilon\tilde{\partial})F = \log\left[1 + \frac{1}{\epsilon}\sum_1^\infty \frac{\mu^{-n} - \lambda^{-n}}{\mu - \lambda}\chi_n(-\epsilon\tilde{\partial})\partial_X F\right]. \tag{2.23}$$

In passing this to limits only the second-order derivatives survive, and one gets the dispersionless differential Fay identity (note that $\chi_n(-\epsilon\tilde{\partial})$ only contributes $-\epsilon\partial_n/n$ here)

$$\sum_{m,n=1}^\infty \mu^{-m}\lambda^{-n}\frac{F_{mn}}{mn} = \log\left(1 - \sum_1^\infty \frac{\mu^{-n} - \lambda^{-n}}{\mu - \lambda}\frac{F_{1n}}{n}\right). \tag{2.24}$$

Although (2.24) only uses a subset of the Plücker relations defining the KP hierarchy, it was shown in [29] that this subset is sufficient to determine KP; hence (2.24) characterizes the function F for DKP. Following [7], we now derive a dispersionless limit of the Hirota bilinear equations (2.12), which we call the dispersionless Hirota equations. We first note from (2.19) and (2.17) that $F_{1n} = nP_{n+1}$ so

$$\sum_1^\infty \lambda^{-n}\frac{F_{1n}}{n} = \sum_1^\infty P_{n+1}\lambda^{-n} = \lambda - P(\lambda). \tag{2.25}$$

Consequently the right-hand side of (2.24) becomes

$$\log\left[\frac{P(\mu)-P(\lambda)}{\mu-\lambda}\right]$$

and for $\mu \to \lambda$ with $\dot{P} := \partial_\lambda P$ we have

$$\log \dot{P}(\lambda) = \sum_{m,n=1}^{\infty} \lambda^{-m-n}\frac{F_{mn}}{mn} = \sum_{j=2}^{\infty}\left(\sum_{n+m=j}\frac{F_{mn}}{mn}\right)\lambda^{-j}. \tag{2.26}$$

Then using the elementary Schur polynomial defined in (2.8) and (2.17), we obtain

$$\begin{aligned}\dot{P}(\lambda) &= \sum_{0}^{\infty}\chi_j(Z_1=0, Z_2, \ldots, Z_j)\lambda^{-j} \\ &= 1+\sum_{1}^{\infty} jP_{j+1}\lambda^{-j-1} = 1+\sum_{1}^{\infty}F_{1j}\lambda^{-j-1}\end{aligned} \tag{2.27}$$

where Z_i, $i \geqslant 2$ are defined by

$$Z_i = \sum_{m+n=i}\frac{F_{mn}}{mn}. \tag{2.28}$$

Thus we obtain the dispersionless Hirota equations

$$F_{1j} = \chi_{j+1}(Z_1=0, Z_2, \ldots, Z_{j+1}). \tag{2.29}$$

These can also be derived directly from (2.12) with (2.20) in the limit $\epsilon \to 0$ or by expanding (2.26) in powers of λ^{-n}. We list here a few entries from such an expansion (cf [7]):

$$\begin{aligned}
&\lambda^{-4}: && \tfrac{1}{2}F_{11}^2 - \tfrac{1}{3}F_{13} + \tfrac{1}{4}F_{22} = 0 \\
&\lambda^{-5}: && F_{11}F_{12} - \tfrac{1}{2}F_{14} + \tfrac{1}{3}F_{23} = 0 \\
&\lambda^{-6}: && \tfrac{1}{3}F_{11}^3 - \tfrac{1}{2}F_{12}^2 - F_{11}F_{13} + \tfrac{3}{5}F_{15} - \tfrac{1}{9}F_{33} - \tfrac{1}{4}F_{24} = 0 \\
&\lambda^{-7}: && F_{11}^2F_{12} - F_{12}F_{13} - F_{11}F_{14} + \tfrac{2}{3}F_{16} - \tfrac{1}{6}F_{34} - \tfrac{1}{5}F_{25} = 0 \\
&\lambda^{-8}: && \tfrac{1}{4}F_{11}^4 - F_{11}F_{12}^2 - F_{11}^2F_{13} + \tfrac{1}{2}F_{13}^2 + F_{12}F_{14} \\
& && \qquad +F_{11}F_{15} - \tfrac{5}{7}F_{17} + \tfrac{1}{16}F_{44} + \tfrac{2}{15}F_{35} + \tfrac{1}{6}F_{26} = 0 \\
&\lambda^{-9}: && F_{11}^3F_{12} - \tfrac{1}{3}F_{12}^3 - 2F_{11}F_{12}F_{13} - F_{11}^2F_{14} + F_{13}F_{14} \\
& && \qquad +F_{12}F_{15} + F_{11}F_{16} - \tfrac{3}{4}F_{18} + \tfrac{1}{10}F_{45} \\
& && \qquad +\tfrac{1}{9}F_{36} + \tfrac{1}{7}F_{27} = 0 \\
&\lambda^{-10}: && \tfrac{1}{5}F_{11}^5 - \tfrac{3}{2}F_{11}^2F_{12}^2 - F_{11}^3F_{13} + F_{12}^2F_{13} + F_{11}F_{13}^2 \\
& && \qquad +2F_{11}F_{12}F_{14} - \tfrac{1}{2}F_{14}^2 + F_{11}^2F_{15} - F_{13}F_{15} - F_{12}F_{16} \\
& && \qquad -F_{11}F_{17} + \tfrac{7}{9}F_{19} - \tfrac{1}{25}F_{55} - \tfrac{1}{12}F_{46} \\
& && \qquad -\tfrac{2}{21}F_{37} - \tfrac{1}{8}F_{28} = 0.
\end{aligned} \tag{2.30}$$

These equations are discussed in various ways below; we will also show the equivalence of the dispersionless Fay differential identity with another formula of a Cauchy kernel in section 3. Note here that for $U = F_{11}$ the first equation in (2.30) is a DKP equation $U_T = 3UU_X + \frac{3}{4}\partial^{-1}U_{YY}$ and other equations in the hierarchy are generated in a similar way.

It is also interesting to note that the dispersionless Hirota equations (2.29) or (2.30) can be regarded as algebraic equations for 'symbols' F_{mn}, which are defined via (2.21), i.e.

$$\mathcal{B}_n := \lambda_+^n = \lambda^n - \sum_1^\infty \frac{F_{nm}}{m}\lambda^{-m}. \tag{2.31}$$

Lemma.1. The symbols satisfy

$$F_{nm} = F_{mn} = Res_P[\lambda^m \mathrm{d}\lambda_+^n]. \tag{2.32}$$

Proof. One need simply observe that

$$\begin{aligned} F_{nm} &= -Res_\lambda[\mathcal{B}_n \mathrm{d}\lambda^m] = -Res_P[\mathcal{B}_n \mathrm{d}\lambda^m] \\ &= -Res_P[\lambda_+^n \mathrm{d}\lambda_-^m] = -Res_P[\lambda^n \mathrm{d}\lambda_-^m] \\ &= Res_P[\lambda^n \mathrm{d}\lambda_+^m] = Res_P[\lambda^m \mathrm{d}\lambda_+^n] = F_{mn}. \end{aligned} \tag{2.33}$$

Here we have used $\lambda_-^m = \lambda^m - \lambda_+^m$ and $Res[\mathrm{d}(ab)] = 0 = Res[\mathrm{d}a\, b + a\, \mathrm{d}b]$ for pseudo-differential or formal Laurent expansions a and b. □

Thus we have the following theorem.

Theorem 1. For λ, P given algebraically as in (2.17), with no *a priori* connection with DKP, and for $\mathcal{B}_n$ defined as in the last equation of (2.31) via a formal collection of the symbols with two indices F_{mn}, it follows that the dispersionless Hirota equations (2.29) or (2.30) are quite simply the polynomial identities between F_{mn}.

In the next section we give a direct proof of this fact, i.e. the F_{mn} defined in (2.31) satisfy the dispersionless Hirota equations, which we designate by (2.29) in what follows.

Now one very natural way of developing DKP begins with (2.17) and (2.18) since eventually the P_{j+1} can serve as universal coordinates (cf here [4] for a discussion of this in connection with topological field theory (TFT)). This point of view is also natural in terms of developing a Hamilton–Jacobi theory involving ideas from the hodograph–Riemann-invariant approach (cf [6, 18, 21, 22] and (4.18) below) and in connecting NKdV ideas to TFT, strings and quantum gravity (cf [10] for a survey of this). It is natural here to work with $Q_n := (1/n)\mathcal{B}_n$ and note that $\partial_n S = \mathcal{B}_n$ corresponds to $\partial_n P = \partial \mathcal{B}_n = n\partial Q_n$. In this connection one often uses different time variables, say $T_n' = nT_n$, so that $\partial_n' P = \partial Q_n$, and $G_{mn} = F_{mn}/mn$ is used in place of F_{mn}. Here, however, we will retain the T_n notation with $\partial_n S = nQ_n$ and $\partial_n P = n\partial Q_n$ since one will be connecting a number of formulae to standard KP notation. Now given (2.17) and (2.18) the equation $\partial_n P = n\partial Q_n$ corresponds to Benney's moment equations and is equivalent to a system of Hamiltonian equations defining the DKP hierarchy (cf [6, 22] and remarks after (2.18); the Hamilton–Jacobi equations are $\partial_n S = nQ_n$ with Hamiltonians $nQ_n(X, P = \partial S)$).

We now have an important formula for the functions Q_n:

Proposition 1 ([22]). The generating function of $\partial_P Q_n(\lambda)$ is given by

$$\frac{1}{P(\mu) - P(\lambda)} = \sum_1^\infty \partial_P Q_n(\lambda)\mu^{-n}. \tag{2.34}$$

Proof. Multiplying (2.17) by $\lambda^{n-1}\partial_P\lambda$, we have

$$\lambda^n\partial_P\lambda = P(\lambda)\lambda^{n-1}\partial_P\lambda + P_2\lambda^{n-2}\partial_P\lambda + \cdots . \tag{2.35}$$

Taking the polynomial part leads to a recurrence relation

$$\partial_P Q_{n+1}(\lambda) = P(\lambda)\partial_P Q_n(\lambda) + P_2\partial_P Q_{n-1}(\lambda) + \cdots + P_n\partial_P Q_1(\lambda) . \tag{2.36}$$

Then noting $\partial_P Q_1 = 1$, $Q_0 = 1$, and summing up (2.36) as follows, we obtain

$$\sum_{n=0}^{\infty}\left(\partial_P Q_{n+1}(\lambda) - P(\lambda)\partial_P Q_n(\lambda) - \sum_{i+j=n} P_{j+1}\partial_P Q_i(\lambda)\right)\mu^{-n}$$

$$= \left(\mu - P(\lambda) - \sum_1^{\infty} P_{j+1}\mu^{-j}\right)\sum_1^{\infty}\partial_P Q_i(\lambda)\mu^{-i} = 1 \tag{2.37}$$

which is just (2.34). □

This is a very important kernel formula which will come up in various ways in what follows. In particular we note

$$\frac{1}{2\pi \mathrm{i}}\oint_{\infty}\frac{\mu^n}{P(\mu) - P(\lambda)}\,\mathrm{d}\mu = \partial_P Q_{n+1}(\lambda) \tag{2.38}$$

which gives a key formula in the Hamilton–Jacobi method for the DKP [22]. Thus it represents a Cauchy kernel and has a version on Riemann surfaces related to the prime form (cf [23]). In fact the kernel is a dispersionless limit of the Fay prime form. Also note here that the function $P(\lambda)$ alone provides all the information necessary for the DKP theory. This will be discussed further in the next section.

3. Solutions and variations

We have already discussed solutions of the dispersionless Hirota equations (2.29) briefly in theorem 1, but we go now to some different points of view. First let us prove the following theorem.

Theorem 2. The kernel formula (2.34) is equivalent to the dispersionless differential Fay identity (2.24).

This will follow from the following lemma.

Lemma 2. Let $\chi_n(Q_1, \ldots, Q_n)$ denote the elementary Schur polynomials defined as in (2.8). Then

$$\partial_P Q_n = \chi_{n-1}(Q_1, \ldots, Q_{n-1}) . \tag{3.1}$$

Proof. We integrate the kernel (2.34) with respect to $P(k)$, and normalize at $\lambda = \infty$ to obtain

$$\frac{1}{P(\lambda) - P(k)} = \frac{1}{\lambda}\exp\left(\sum_1^{\infty} Q_n(k)\lambda^{-n}\right) = \sum_1^{\infty}\chi_{m-1}(Q_1, \ldots, Q_{m-1})\lambda^{-m} . \tag{3.2}$$

Equating this with (2.34) gives the result. □

Proof of theorem 2. Using equation (2.21), the left-hand side of (2.24) can be written as

$$\text{LHS} = \sum_1^\infty \frac{1}{n\lambda^n}\left[\sum_1^\infty \frac{F_{mn}}{m\mu^m}\right] = \sum_1^\infty \frac{1}{n\lambda^n}[\mu^n - \mathcal{B}_n(\mu)]$$

$$= \sum_1^\infty \frac{1}{n}\left(\frac{\mu}{\lambda}\right)^n - \sum_1^\infty \frac{1}{n\lambda^n}\mathcal{B}_n(\mu) = -\log\left(1-\frac{\mu}{\lambda}\right) - \sum_1^\infty \frac{Q_n(\mu)}{\lambda^n}\,. \tag{3.3}$$

On the other hand, the right-hand side of (2.24) is calculated as indicated in (2.25) and remarks thereafter. Thus

$$\text{RHS} = \log\left[1 - \frac{1}{\lambda-\mu}\sum_1^\infty\left(\frac{1}{\lambda^n} - \frac{1}{\mu^n}\right)\frac{F_{1n}}{n}\right]$$

$$= -\log(\lambda-\mu) + \log\left[\left(\lambda - \sum_1^\infty \frac{F_{1n}}{n\lambda^n}\right) - \left(\mu - \sum_1^\infty \frac{F_{1n}}{n\mu^n}\right)\right]$$

$$= -\log(\lambda-\mu) + \log[P(\lambda) - P(\mu)]\,. \tag{3.4}$$

This implies

$$\log[P(\lambda) - P(\mu)] = \log\lambda - \sum_1^\infty Q_n(\mu)\lambda^{-n} \tag{3.5}$$

which leads to (3.2). Then lemma 2 implies the assertion. □

Theorem 2 implies that the dispersionless Hirota equations (2.29) can be derived from the kernel formula (2.34) which is a direct consequence of the definition of F_{mn} in (2.31). Thus theorem 2 gives a direct proof of theorem 1. □

We will now express the $\chi_n(Q_1, \ldots, Q_n)$ as polynomials in $Q_1 = P$ with the coefficients given by polynomials of P_{j+1}. First we have:

Lemma 3. One can write

$$\chi_n = \det\begin{bmatrix} P & -1 & 0 & 0 & 0 & \cdots & 0 \\ P_2 & P & -1 & 0 & 0 & \cdots & 0 \\ P_3 & P_2 & P & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_n & P_{n-1} & \cdots & P_4 & P_3 & P_2 & P \end{bmatrix} = \partial_P Q_{n+1}\,. \tag{3.6}$$

Proof. In terms of $\chi_n(Q_1, \ldots, Q_n) = \partial_P Q_{n+1}$ the recursion relation (2.36) can be written in the form $\chi_n = P\chi_{n-1} + P_2\chi_{n-2} + \cdots + P_n\chi_0$. It is easy to derive the determinant expression from this form. □

This leads to a rather evident fact, which we express as a proposition because of its importance. Thus:

Proposition 2. The F_{mn} can be expressed as polynomials in $P_{j+1} = F_{1j}/j$.

Proof. $\partial_\lambda Q_n = \chi_{n-1}(Q_1; P_2, \ldots, P_{n-1})\partial_\lambda Q_1$ ($Q_1 = P$). We put this together with $Q_n = \mathcal{B}_n/n$ in (2.21) and P in (2.17) to arrive at the conclusion (cf also remark 1). □

Corollary 1. The dispersionless Hirota equations can be solved totally algebraically via $F_{mn} = \Phi_{mn}(P_2, P_3, \ldots, P_{m+n})$ where Φ_{mn} is a polynomial in the P_{j+1}. Thus the $F_{1n} = nP_{n+1}$ are generating elements for the F_{mn}.

This is of course evident from the dispersionless differential Fay identity (2.24). The point here however is to give explicit formulae of F_{mn} in terms of the elementary Schur polynomials χ_n in (3.6).

Remark 1. One can also arrive at this polynomial dependence via the residue formulae of lemma 1. As an adjunct to the proof of proposition 2 we note now the following explicit calculations. Recall that $Q_1 = P = \lambda - \sum_1^\infty P_{j+1}\lambda^{-j}$ and generally $Q_n = \lambda^n/n - \sum_1^\infty G_{jn}\lambda^{-j}$ ($G_{jn} := F_{jn}/jn$). Thus from $Q_2 = \lambda_+^2/2 = P^2/2 + P_2$ we get

$$\frac{1}{2}\left(\lambda - \sum_1^\infty \frac{P_{j+1}}{\lambda^j}\right)^2 + P_2 = \frac{\lambda^2}{2} - \sum_1^\infty \frac{G_{2j}}{\lambda^j}. \tag{3.7}$$

Writing this out yields

$$G_{2m} = P_{2+m} - \frac{1}{2}\sum_{j+k=m} P_{j+1}P_{k+1}. \tag{3.8}$$

This process can be continued with $Q_3 = \lambda_+^3/3 = P^3/3 + P_2P + P_3$, etc.

It is also interesting to note the following relations.

Proposition 3. For any $n \geqslant 2$,

$$\chi_n(-Q_1, \ldots, -Q_n) = -P_n. \tag{3.9}$$

Proof. We obtain (cf equation (2.8))

$$\left(\sum_0^\infty \frac{\chi_n(Q)}{\lambda^n}\right)\left(\sum_0^\infty \frac{\chi_m(-Q)}{\lambda^m}\right) = \sum_0^\infty \frac{1}{\lambda^p}\sum_0^p \chi_{p-k}(Q)\chi_k(-Q) = 1. \tag{3.10}$$

Hence $\sum_0^p \chi_{p-k}(Q)\chi_k(-Q) = 0$ for $p \geqslant 1$ which implies $\chi_k(-Q) = -P_k$ for $k \geqslant 2$ from the recursion relation (2.36). □

Note from $P_{n+1} = F_{1n}/n$ that the equations (3.9) give another representation of the dispersionless Hirota equations (2.29).

Remark 2. Formulae such as (3.6) and the statement in proposition 2 indicate that in fact DKP theory can be characterized using only elementary Schur polynomials since these provide all the information necessary for the kernel (2.34) or equivalently for the dispersionless differential Fay identity. This amounts also to observing that in the passage from KP to DKP only certain Schur polynomials survive the limiting process $\epsilon \to 0$. Such terms involve second derivatives of F and these may be characterized in terms of Young diagrams with only vertical or horizontal boxes. This is also related to the explicit form of the hodograph transformation where one needs only $\partial_P Q_n = \chi_{n-1}(Q_1, \ldots, Q_{n-1})$ and the P_{j+1} in the expansion of P (cf equation (4.18) here).

4. Connections to D-bar

It was shown in [6, 7, 17, 31] how inverse scattering information is connected to the dispersionless theory for KdV and some other situations (Benney equations and vector nonlinear Schrödinger equations for example). We will see here that although such connections seem generally not to be expected, nevertheless, one can isolate D-bar data for S and P in the dispersionless NKdV situations leading to an expression for the generating elements $F_{1n} = nP_{n+1}$ which can be useful in computation. The technique also indicates another role for the Cauchy-type kernel $1/(P(\lambda) - P(\mu))$ in (2.34). Thus as background consider KdV and the scattering problem in the standard form (cf [8])

$$u_t = u''' - 6uu' \qquad \psi'' - u\psi = -k^2\psi\,. \tag{4.1}$$

Let $\psi_\pm(k, x) \sim \exp(\pm ikx)$ as $x \to \pm\infty$ be Jost solutions with scattering data, T and R, determined via

$$T(k)\psi_-(k, x) = R(k)\psi_+(k, x) + \psi_+(-k, x)\,. \tag{4.2}$$

Setting $\psi_- = \exp(-ikx + \phi(k, x))$ we have from (4.1)

$$v := -\partial \log(\psi_-) = ik - \phi' \qquad \phi'' - 2ik\phi' + \phi'^2 = u\,. \tag{4.3}$$

One seeks expansions

$$\phi' = \sum_1^\infty \frac{\phi_n}{(ik)^n} \qquad v = ik + \sum_1^\infty \frac{v_n}{(ik)^n} \tag{4.4}$$

entailing $\phi_n = -v_n$. Here for example one can assume $\int_{-\infty}^{\infty}(1 + x^2)|u|\,dx < \infty$ or $u \in \mathcal{S} =$ Schwartz space for convenience so that all of the inverse scattering machinery applies. Then $T(k)$ will be meromorphic for $\mathrm{Im}(k) > 0$ with (possibly) a finite number of simple poles at $k_j = i\beta_j$ ($\beta_j > 0$), $|R(k)|$ is small for large $|k|$, $k \in \mathbb{R}$, and $\log(T) = \sum_0^\infty c_{2n+1}/k^{2n+1}$ where c_n can be written in terms of the β_j and the normalization of the wavefunction $\psi_\pm$. For $x \to \infty$, $\mathrm{Im}(k) > 0$, $\psi_- \exp(ikx) \to 1/T$ from (4.2), and taking logarithms $\phi(k, \infty) = -\log(T)$ which implies

$$-\log(T) = \sum_1^\infty \frac{1}{(ik)^n} \int_{-\infty}^{\infty} \phi_n\,dx \tag{4.5}$$

leading to $\int_{-\infty}^{\infty} \phi_{2m}\,dx = 0$ with $i^{2m}c_{2m+1} = \int_{-\infty}^{\infty} \phi_{2m+1}\,dx$. Hence the scattering information ($\sim c_{2n+1}$, arising from x-asymptotics, given via R and T) is related to the k-asymptotics ϕ_n of the wavefunction ψ_-. This kind of connection between space asymptotics (or spectral data, which are generated by these asymptotic conditions and forced by boundary conditions on u here) and the spectral asymptotics of the wavefunction is generally more complicated and we refer to a formula in the appendix to [6] describing the Davey–Stewartson (DS) situation

$$-2\pi i\chi_{n+1} = \int_\Omega\int \zeta^n \bar{\partial}_\zeta \chi\,d\zeta \wedge d\bar{\zeta} \tag{4.6}$$

where $\bar{\partial}_\zeta \chi$ is D-bar data and the wavefunction ψ has the form $\psi = \chi\exp(\Xi)$ with $\chi = 1 + \sum_1^\infty \chi_j\lambda^{-j}$ for $|\lambda|$ large (this is in a matrix form). The potentials in (4.6) occur in χ_1. The D-bar data, or departure from analyticity of χ, correspond to spectral data in a sometimes complicated way and we refer the reader to [1, 5, 6, 9, 11, 15, 19, 24, 26, 27] for more on this.

Now one expects some connections between inverse scattering for KdV and the DKdV theory since $\epsilon x = X$ means, e.g., $x \to \pm\infty \sim \epsilon \to 0$ for fixed X. Now from (4.2), we have

$$\frac{1}{T} = \frac{1}{2\mathrm{i}k} W(\psi_+, \psi_-) := \frac{1}{2\mathrm{i}k}(\psi_+'\psi_- - \psi_+\psi_-') . \tag{4.7}$$

Given $\psi_+ \sim \psi = \exp(S/\epsilon)$ and $\psi_- \sim \psi^* = \exp(-S/\epsilon)$ (cf [6, 7]) one sees that $W(\psi_+, \psi_-) \to 2\partial S/\partial X = 2P$, and from (2.17) with $\lambda = \mathrm{i}k$

$$\frac{1}{T} = \frac{P}{\mathrm{i}k} = 1 - \sum_1^\infty \frac{P_{n+1}}{(\mathrm{i}k)^{n+1}} . \tag{4.8}$$

Now $\log|T|$ and $\arg(R/T)$ have natural roles as action angle variables (cf [7]) and from $R = W(\psi_-(k,x), \psi_+(-k,x))/W(\psi_+, \psi_-)$ one obtains

$$R = -\left(\frac{P(-k) + P(k)}{2P(k)}\right) \exp\left(\frac{S(k) - S(-k)}{\epsilon}\right) . \tag{4.9}$$

(We have suppressed the slow variables T_n.) Thus the scattering data, which are considered to be averaged over the fast variables, now depend on the slow variables (in particular, see (4.5) and (4.8)). This corresponds to a dynamics of the Riemann surface determined by the scattering problem (the Whitham averaging approach). The exponential terms in (4.9) could in principle give problems here so we consider this in the spirit of [6, 7, 17]. Thus consider $\psi_- = \exp(-\mathrm{i}kx + \phi)$ with $\phi(x) = \int_{-\infty}^x \phi'(\xi)\,\mathrm{d}\xi$. Now ψ_- is going to have the form $\psi_- = \exp(-S(X,k)/\epsilon)$ and we can legitimately expect $-S/\epsilon = -\mathrm{i}kX/\epsilon + \phi(X/\epsilon, k)$. For this to make sense let us write (note $\epsilon x = X$, $\epsilon\xi = \Xi$)

$$\phi(X/\epsilon, k) = \int_{-\infty}^{X/\epsilon} \phi'(\xi)\,\mathrm{d}\xi = \frac{1}{\epsilon}\int_{-\infty}^{X} \phi'(\Xi/\epsilon)\,\mathrm{d}\Xi = \frac{1}{\epsilon}(\mathrm{i}kX - S(X,k)) . \tag{4.10}$$

Thus for $\phi(X/\epsilon, k)$ to equal $\epsilon^{-1}(\mathrm{i}kX - S(X,k))$ we must have

$$\int_{-\infty}^{X} \phi'(\Xi/\epsilon)\,\mathrm{d}\Xi = \mathrm{i}kX - S(X,k) \tag{4.11}$$

where $\phi' = \partial\phi/\partial x$. This says that $\phi'(\Xi/\epsilon) = f(\Xi/\epsilon) = \tilde{f}(\Xi) + \mathrm{O}(\epsilon)$ which is reasonable in the same spirit that $u_n(T_j/\epsilon) = U_n(T_j) + \mathrm{O}(\epsilon)$ was reasonable before. Thus we have

$$P = \frac{\partial S}{\partial X} = \mathrm{i}k - \phi' = v \qquad P^2 - U = -k^2 \tag{4.12}$$

which corresponds to a dispersionless form of (4.3) in new variables.. Note here that $\phi'' = \epsilon\partial\phi'/\partial X \to 0$ as $\epsilon \to 0$. Thus we have:

Proposition 4. The manipulation of variables x, X in (4.10) and (4.11) is consistent and mandatory. It shows that scattering information (expressed via ϕ for example) is related to the dispersionless quantity S or P.

Further, we have isolated D-bar data ($\bar{\partial}_k S$) for S since for $P = \sqrt{U - k^2}$ one can write for arbitrary X_0 (cf equation (4.11))

$$\mathrm{i}kX - S(X,k) = \int_{-\infty}^{X} (\mathrm{i}k - P(\Xi,k))\,\mathrm{d}\Xi . \tag{4.13}$$

Replace $-\infty$ by X_0 for large negative X_0 to then get $S = \int_{X_0}^X P\,\mathrm{d}\Xi + \mathrm{i}kX_0$. Thus one expects a pole for $|k| \to \infty$ plus D-bar data along $(-\sqrt{U}, \sqrt{U})$. This is in the spirit of

[17, 18] where it is phrased differently. Observe that the expression $P = \sqrt{U-k^2}$ places us on a two-sheeted Riemann surface with a cut along $(-\sqrt{U}, \sqrt{U})$ in the k-plane. Going around say $\sqrt{U}$ on the $+$ sheet one has opposite sides of the cut $P_+ = \mathrm{i}\sqrt{k^2-U}$ and $P_- = -\mathrm{i}\sqrt{k^2-U}$, so $P_+ - P_- = \Delta P = 2\mathrm{i}\sqrt{k^2-U}$ which corresponds to D-bar data. This then leads to

$$\Delta S = S_+ - S_- = \int_{X_0}^{X} \Delta P \,\mathrm{d}\Xi = 2\int_{X_0}^{X} \sqrt{U(\Xi)-k^2}\,\mathrm{d}\Xi = 2\,\mathrm{Re}\, S_+ . \tag{4.14}$$

Note here that $P_\pm$ is real on the cut with $P_- = -P_+$ (cf remark 3 for some general comments).

Consider next the situation of [6, 18, 22] with a reduction of DKP to

$$\Lambda := \lambda^N = P^N + a_0 P^{N-2} + \cdots + a_{N-2} \tag{4.15}$$

which is called DNKdV reduction. Assume $\Lambda(P)$ has N distinct real zeros $P_1 > P_2 > \cdots > P_N$ with $N-1$ interwoven turning points $\Lambda_k = \Lambda(\tilde{P}_k)$, $P_{k+1} > \tilde{P}_k > P_k$. Assume, as $X \to -\infty$, these Riemann invariants $\Lambda_k \to 0$ monotonically so $\Lambda \to P^N$ with $(P-\lambda) \to 0$. Such situations are considered in [18] and techniques from [22] are adapted to produce formulae of the type ($W_m = \partial_P \mathcal{B}_m$)

$$S(X,\mu) - P(X,\mu)X - \mathcal{B}_m(P(X,\mu), X)T_m = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{S(X,\lambda)\partial_\lambda P(X,\lambda)}{P(X,\lambda)-P(X,\mu)}\,\mathrm{d}\lambda \tag{4.16}$$

$$\begin{aligned}\frac{\partial S}{\partial\mu} - \frac{\partial P}{\partial\mu}(X + W_m(P,X)T_m) &= -\frac{1}{2\pi\mathrm{i}}\frac{\partial P}{\partial\mu}\oint_\Gamma \frac{S(X,\lambda)\partial_\lambda P(X,\lambda)}{(P(X,\lambda)-P(X,\mu))^2}\,\mathrm{d}\lambda \\ &= -\frac{\partial_\mu P}{2\pi\mathrm{i}}\oint_\Gamma \frac{\partial_\lambda S(X,\lambda)}{P(X,\lambda)-P(X,\mu)}\,\mathrm{d}\lambda \\ &= \frac{\partial_\mu P}{\pi}\int_{\Gamma_+} \frac{\partial_\lambda \,\mathrm{Im}\, S(X,\lambda)}{P(X,\lambda)-P(X,\mu)}\,\mathrm{d}\lambda .\end{aligned} \tag{4.17}$$

We see here the emergence of the Cauchy-type kernel from (2.34) in an important role. At $\lambda_k := \lambda(\tilde{P}_k)$ where S_μ is bounded and $P_\mu \to \infty$ (4.17) yields Tsarev-type generalized hodograph formulae (cf [18]). Thus such a formula is

$$X + W_m(\tilde{P}_k(X), X)T_m = -\frac{1}{\pi}\int_{\Gamma_+} \frac{\partial_\lambda \,\mathrm{Im}\, S(X,\lambda)}{P(X,\lambda)-\tilde{P}_k(X)}\,\mathrm{d}\lambda . \tag{4.18}$$

Here one has Riemann invariants $\lambda_k = \lambda(\tilde{P}_k)$ where $\partial_P \Lambda = 0$ and there is a collection L of finite cuts through the origin of angles $k\pi/N$ $(1 \leqslant k \leqslant N-1)$ in the λ plane with branch points λ_k. One takes Γ to be a contour encircling the cuts clockwise (not containing μ) and sets $\Gamma = \Gamma_- - \Gamma_+$ where $(+)$ refers to the upper half-plane. It turns out that $S|_{\Gamma_+} = \bar{S}|_{\Gamma_-}$ and the contours can be collapsed onto the cuts to yield the last integral in (4.17) (see [18] for pictures). By reorganizing the terms in the integrals one can now express integrals such as (4.17) in terms of D-bar data of P on the cuts. This is straightforward, but details for this and other constructions below will be spelled out in [11] for completeness. Thus P and S are analytic in λ for finite λ except on the cuts L where there is a jump discontinuity ΔP (yielding ΔS by integration in X as in (4.14)). We have seen for DKdV that $\Delta P = 2\sqrt{U-k^2}$ on $(-\sqrt{U}, \sqrt{U})$ and other DNKdV situations are similar. Moreover there is no need to restrict ourself to one time variable T_m in (4.17), or for that matter to DNKdV. Indeed the techniques that lead to formulae of the type (4.17) are based on [22] (cf also [6]), and apply equally well to DKP provided the D-bar data $\bar{\partial}P$ for DKP lie in a bounded set Ω (cf remarks below). In this respect, concerning DKP, one first notes that the transformation $t \to -t$, $U \to -U$ sends DKP-1 to DKP-2 so one expects that any D-bar data for S or P will be the same. Secondly, we know from [6, 22] that for large $|\lambda|$ there will be formulae of the type (2.34). Let us assume that $\bar{\partial}P$ (and hence $\bar{\partial}S$) is non-trivial only for a region Ω where, say, $|\lambda| \leqslant M < \infty$, and let Γ be a contour enclosing $|\lambda| \leqslant M$. Then without regard for the nature of such data (Riemann–Hilbert data, poles, simple non-analyticity, etc) one can in fact derive formulae (4.17) following [6, 18, 22]. We cannot collapse on cuts as in the last equation of (4.17) but we can think of the other formulae in (4.17) as integrals over D-bar data $\bar{\partial}P$ or $\bar{\partial}S$. It remains open to describe D-bar data for the DKP situations, however. The idea of some kind of limit of spiked cut collections arises, but we have not investigated this. In summary we can state:

Proposition 5. For DNKdV (or DKP with given bounded D-bar data) one can write

$$S(X,\mu) = P(\mu)X + \sum_2^\infty \mathcal{B}_n(P(\mu), X)T_n - \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{S(X,\lambda)\partial_\lambda P(X,\lambda)}{P(X,\lambda) - P(X,\mu)}\,\mathrm{d}\lambda \tag{4.19}$$

where Γ encircles the cuts (or D-bar data) clockwise. The determination of D-bar data here is to be made via analysis of the polynomial λ^N in (4.15) for DNKdV and in this situation one can again collapse (4.17) to the cuts.

Remark 3. Now perhaps the main point of this has been to show that S can be characterized via D-bar data of S or P. In general we do not expect D-bar or scattering data for NKdV or KP to give D-bar data for S. The case of KdV is probably exceptional here and the situation of [17, 31] where spectral data for a system of nonlinear Schrödinger equations is related to the Benney or DKP hierarchy involves a different situation (the spectral data are not related to KP). Furthermore, since KP-1 and KP-2 have vastly different spectral or D-bar properties and both pass to the same DKP, one does not expect the spectral data to play a role. We note also that spectral data are created by the potentials via asymptotic conditions for example (and vice versa) whereas D-bar data for DNKdV, arising from the polynomial λ^N, are a purely algebraic matter.

There is an interesting way in which D-bar data for P or S can be exploited. Thus, given that S has D-bar data as indicated above in a bounded region, one can say for $|\lambda|$ large and with some analytic function A of λ that

$$\begin{aligned} S &= A + \frac{1}{2\pi \mathrm{i}} \int\!\!\int \frac{\bar{\partial}_\zeta S}{\zeta - \lambda}\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \\ &= A - \frac{1}{2\pi \mathrm{i}} \sum_0^\infty \frac{1}{\lambda^{j+1}} \int\!\!\int \zeta^j \bar{\partial}_\zeta S\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \end{aligned} \tag{4.20}$$

and from (2.19) we get

$$\partial_j F = \frac{j}{2\pi \mathrm{i}} \int\!\!\int \zeta^{j-1} \bar{\partial}_\zeta S\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\,. \tag{4.21}$$

This is very useful information about F_j, since from the computation of (4.21) one could in principle compute all of the functions F_{ij}, for example. In fact, for DKdV, and possibly some other DNKdV situations, one can obtain a direct formula for the F_{1j}, which we know to generate all the F_{ij} via lemma 1. Thus for DKdV we know from (4.14) that $\Delta S = \int_{X_0}^X \Delta P\,\mathrm{d}X' = 2\int_{X_0}^X \sqrt{U(X',T) - k^2}\,\mathrm{d}X'$ on the cut $L = (-\sqrt{U}, \sqrt{U})$ in the

k plane, and $\bar{\partial}S = \Delta S(\mathrm{i}/2)\delta_L$ where δ_L is a suitable delta function on L. Adjusting variables ($\zeta^2 = -k^2$ etc) one can then write

$$\partial F_j = -\frac{j(\mathrm{i})^{j-1}}{\pi}\partial\int_{-\sqrt{U}}^{\sqrt{U}} k^{j-1}\int_{X_0}^{X}\sqrt{U(X',T)-k^2}\,\mathrm{d}X'\,\mathrm{d}k\,. \tag{4.22}$$

The only X dependence here is visible and one can differentiate under the integral sign in (4.22). Since all $F_{1\ 2m} = 0$ by the residue formula (2.32), for example we take $j = 2n-1$ to obtain

$$F_{1\ 2n-1} = (-1)^n\frac{2n-1}{\pi}\int_{-\sqrt{U}}^{\sqrt{U}} k^{2n-2}\sqrt{U-k^2}\,\mathrm{d}k =$$

$$= (-1)^n\frac{(2n-1)U^n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{2n-2}\theta\cos^2\theta\,\mathrm{d}\theta = (-1)^n\left(\frac{U}{2}\right)^n\prod_1^n\frac{2l-1}{l} \tag{4.23}$$

(here $k = a\sin\theta$ and $a^2 = U$). We summarize this in (cf [11] for more detail):

Theorem 3. For any situation with bounded D-bar data for S one can compute F_j from (4.21). In the case of DKdV with ΔS as indicated one can use (4.23) to determine directly the $F_{1\,2n-1}$ which will generate all the functions F_{ij}. Generally, if $\bar{\partial}S = \int_{X_0}^{X}\bar{\partial}P(X',T,\zeta,\bar{\zeta})\,\mathrm{d}X'$ (as in (4.22)), then from (4.21) $F_{1j} = (j/2\pi\mathrm{i})\int\int\zeta^{j-1}\bar{\partial}P\,\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}$, and formally, $\partial_n\partial_j F = (j/2\pi\mathrm{i})\int\int\zeta^{j-1}\bar{\partial}\mathcal{B}_n\,\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}$ since formally $\partial_n\bar{\partial}S = \bar{\partial}\partial_n S = \bar{\partial}\mathcal{B}_n$.

Remark 4. The calculations using (4.22) in fact agree with the determination of $F_{1\ 2n-1}$ from residue calculations as in lemma 1 (note here $2U_2 = -U$ when adjusting notation between sections 2 and 4). In general one may have some difficulties in determining $\bar{\partial}\mathcal{B}_n$ or $\bar{\partial}P$ for example, so the formulae (4.22) and (4.23) may be exceptional in this regard. However, for DKdV we have checked the validity of these formal integrals for F_{nj} involving $\bar{\partial}\mathcal{B}_n$ for a number of terms (e.g. F_{31}, F_{51}, F_{33}, F_{35}, and F_{53}) against the results of residue calculations. Here one can write for n odd $\bar{\partial}\mathcal{B}_n = \bar{\partial}\zeta_+^n = \Delta\zeta_+^n = P^{-1}\zeta_+^n\Delta P$ and one gets for $n = 2k-1,\ j = 2m-1$

$$F_{nj} = (-1)^m\frac{2(2m-1)U^{k+m-1}}{\pi}\int_0^{\frac{\pi}{2}}\sin^{2m-2}\theta\,\hat{\zeta}_+^{2k-1}(\cos\theta)\cos\theta\,\mathrm{d}\theta \tag{4.24}$$

where $a^{2k-1}\hat{\zeta}_+^{2k-1}(\cos\theta) = \zeta_+^{2k-1}(P)$ for $P = a\cos\theta$ ($a^2 = U$). Let us mention also that one can easily write down the tables of F_{ij} for DNKdV from the table (2.30). Thus for DNKdV we have a reduction of the table (2.30) given via $F_{Nm} = F_{mN} = 0$ for all m.

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